Application of the Fokker-Planck equation to particle-beam injection into e^- storage rings

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Nonlinear forces in the longitudinal accelerating field—or in the transverse magnetic fields—lead to filamentation of the injected emittance and to the decoherence of the center-of-mass motion. The dynamics of the particle distribution function in the presence of synchrotron radiation is governed by the Fokker-Planck equation. We derive the time evolution of the distribution function after injection as an approximate solution to the Fokker-Planck equation. The approximation assumes the injected emittance to be considerably larger than the equilibrium emittance, which is fulfilled for a certain class of storage rings, e.g., damping rings. In the limit of no quantum excitation, this distribution function will then be an exact solution. Higher moments of the distribution can be expressed in combinations of elementary functions and agree very well with multiparticle simulations.

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INTRODUCTION

Injection of a bunched beam into the periodic structure of a storage ring may lead to the formation of filaments in phase space [1]. It is generally assumed that, after some relaxation time, this filamentary structure can be described by a smoothly varying distribution function that gradually approaches equilibrium.

In Ref. [2], the time evolution of the distribution function—after mismatched (the betatron functions of the injection beam ellipsoid and the lattice are different) or off-axis injection—was analyzed by means of the Vlasov equation. The influence of nonlinear fields was approximated by an averaged Hamiltonian that depends only on the action variable. Using this Hamiltonian, the Vlasov equation could be solved exactly.

In order to describe the effects of injection transients for a larger time period than a small fraction of the damping time, the effect of synchrotron radiation on the beam has to be taken into account. In this paper we derive the time evolution of the distribution function as an exact solution of the Fokker-Planck equation in the cases of (a) only linear fields; and (b) nonlinear fields and damping, but no quantum excitation.

In addition we discuss an approximate solution to the Fokker-Planck equation where nonlinear fields, damping, and quantum excitation are taken into account. The approximation assumes the injection emittance is much larger than the equilibrium emittance—this assumption is typically fulfilled in damping rings.

Due to the relatively simple form of the distribution function, first and second moments may be derived in closed expressions. These relations are then compared to results of multiparticle simulations, where radiation damping and the effect of quantum excitation were included.

I. TIME EVOLUTION OF THE DISTRIBUTION FUNCTION NEGLECTING NONLINEAR FIELDS

In this section, we study the time evolution of the distribution function in phase space after mismatched or off-axis injection into a periodic structure. Neglecting nonlinear fields, the single-particle motion may be described by the Hamiltonian

$$H_0(\xi,\eta)=\frac{\nu_0}{2R}(\xi^2+\eta^2)\ .$$

The transformation to the measurable transverse (x,p) and longitudinal (ϵ,z) coordinates is given by

Longitudinal		Transverse
$\epsilon\left[\frac{\alpha R}{\nu_{s0}}\right]^{1/2}=\epsilon\left[\frac{\sigma_{z\infty}}{\sigma_{\epsilon_\infty}}\right]^{1/2}$	$\Longleftarrow\xi\Longrightarrow$	$\frac{x}{\sqrt{\beta}}$
$z\left[\frac{\nu_{s0}}{\alpha R}\right]^{1/2}=z\left[\frac{\sigma_{\epsilon\infty}}{\sigma_{z\infty}}\right]^{1/2}$	$\Longleftarrow\eta\Longrightarrow$	$\frac{\alpha x+\beta p}{\sqrt{\beta}}$
ν_{s0}	$\Longleftarrow\nu_0\Longrightarrow$	ν_{x0}

(1)

where $\sigma_{z\infty},\sigma_{\epsilon\infty}$ denote the bunch length and the energy spread at equilibrium, and α in the longitudinal plane denotes the momentum compaction, whereas α,β in the transverse plane are the Twiss parameters at a fixed position in the ring [3]. The tunes in the longitudinal and in the transverse plane are denoted by ν_{s0} and ν_{x0}. It is useful to be able to work with action-angle variables. We introduce

$$\eta=\sqrt{2I}\,\cos(\phi)\ ,\quad \xi=\sqrt{2I}\,\sin(\phi)\ . \tag{2}$$

With these variables, the Hamiltonian reduces to

$$H_0(I)=\frac{\nu_0}{R}I\ .$$

Electrons receive energy from the accelerating cavities, and lose it again due to synchrotron radiation. To describe this fluctuating radiation process, a stochastic term has to be added to the equations of motion leading to a set of stochastic differential equations [4]. The dynamics of the phase-space particle distribution $\Psi(\phi,I,t)$ is then described by the Fokker-Planck equation. From Ref. [5], we have

$$\tau\Psi_t = 2\Psi + 2(I+\sigma)\Psi_I + 2\sigma I\Psi_{II} - \tau\omega_0\Psi_\phi + \tfrac{1}{2}\sigma\frac{1}{I}\Psi_{\phi\phi} , \quad (3)$$

where the subscripts denote partial differentiation. The oscillation frequency ω_0 is equal to the tune times 2π divided by the revolution time; τ is the damping time; and σ is related either to the transverse equilibrium emittance or, in the longitudinal case, to the product of bunch length and energy spread:

Longitudinal		Transverse
$\sigma_{\epsilon\infty}\sigma_{z\infty}$	$\Longleftarrow\sigma\Longrightarrow$	$\epsilon_{x\infty}$
$\sigma_{\epsilon 0}\sigma_{z0}$	$\Longleftarrow\sigma_0\Longrightarrow$	ϵ_{x0} .

(4)

In analogy to σ, we introduce σ_0, the corresponding term at injection. Before we go on to investigate possible solutions of the Fokker-Planck equation, we want to parametrize the distribution function at injection. We assume a Gaussian distribution both in longitudinal and in transverse phase space. Figure 1 displays the phase-space portrait of three different distributions at injection.

A. Mismatched beam injected on axis

For the moment, we consider the case shown in Fig. 1(a): the centered distribution function where the center of mass of the distribution coincides with the origin of phase space.

In the transverse measurable coordinates (x,p), we parametrize the mismatched injected distribution as an ellipse with $\alpha_0,\beta_0,\epsilon_{x0}$. In the longitudinal case, we assume for simplicity that the injected ellipse is upright; i.e., the major axis of the ellipse is aligned with one of the ξ,η axes. Then the injected longitudinal ellipse is described sufficiently by the bunch length σ_{z0} and energy spread $\sigma_{\epsilon 0}$ of the incoming beam. Using Eq. (1), at the moment of injection we obtain the distribution function in the variables (ξ,η) (see also Ref. [2]) as

Longitudinal		Transverse
$\dfrac{1}{2\pi\sigma_{z0}\sigma_{\epsilon 0}}e^{\{-C_0\xi^2+2A_0\xi\eta+B_0\eta^2\}/2\sigma_{z0}\sigma_{\epsilon 0}}$	$\Longleftarrow\Psi_0(t=0)\Longrightarrow$	$\dfrac{1}{2\pi\epsilon_{x0}}e^{\{-C_0\xi^2+2A_0\xi\eta+B_0\eta^2\}/2\epsilon_{x0}}$
$\dfrac{1}{g}\equiv\dfrac{\sigma_{x\infty}\sigma_{\epsilon 0}}{\sigma_{x0}\sigma_{\epsilon\infty}}$	$\Longleftarrow B_0\Longrightarrow$	$\dfrac{\beta_0}{\beta}$
0	$\Longleftarrow A_0\Longrightarrow$	$\alpha_0-\dfrac{\beta_0\alpha}{\beta}$
$g=\dfrac{\sigma_{z0}\sigma_{\epsilon\infty}}{\sigma_{z\infty}\sigma_{\epsilon 0}}$	$\Longleftarrow C_0\Longrightarrow$	$\dfrac{A_0^2+1}{B_0}$

(5)

where α,β denotes the Twiss parameters of the ring at the injection point. With $g=1$, the longitudinal distribution appears circular in phase space. For example, the longitudinal distribution of an electron bunch injected into the Stanford linear collider (SLC) damping ring is described by $g\approx\frac{1}{25}$. With Eq. (2), the injected distribution function in action-angle variables is given by

$$\Psi_0(t=0)=\frac{1}{2\pi d_0}\exp\{-I[b_0+c_0\cos(2\phi-2\bar{\phi})]\} , \quad (6)$$

with

Longitudinal		Transverse
0	$\Longleftarrow\tan(2\bar{\phi})\Longrightarrow$	$-2A_0B_0/(1+A_0^2-B_0^2)$
$\sigma_{\epsilon 0}\sigma_{z0}$	$\Longleftarrow d_0\Longrightarrow$	ϵ_{x0}
$(g^2+1)/(2g\sigma_{z0}\sigma_{\epsilon 0})$	$\Longleftarrow b_0\Longrightarrow$	$(1+A_0^2+B_0^2)/(2B_0\epsilon_{x0})$
$(1-g^2)/(2g\sigma_{z0}\sigma_{\epsilon 0})$	$\Longleftarrow c_0\Longrightarrow$	$-\sqrt{b_0^2-1/\epsilon_{x0}^2}$.

(7)

We expect the injected ellipse to start to rotate in phase space. From this point of view, we extrapolate from Eq. (6) the assumed time evolution of the distribution function:

$$\Psi_0(t)=\frac{1}{2\pi d(t)}\exp\{-I[b(t)+c(t)\cos(2\Omega)]\} , \quad (8)$$

with

$$\Omega=\phi-\omega_0 t-\bar{\phi} ,$$

where the unknown functions $d(t)$, $b(t)$, and $c(t)$ have to be determined from Eq. (3). We realize that we may rearrange the exponent of the distribution function and write Eq. (8) as

$$\Psi_0(t)=\frac{1}{2\pi d(t)}\exp\{-I[u(t)\cos^2(\Omega)+v(t)\sin^2(\Omega)]\} ,$$

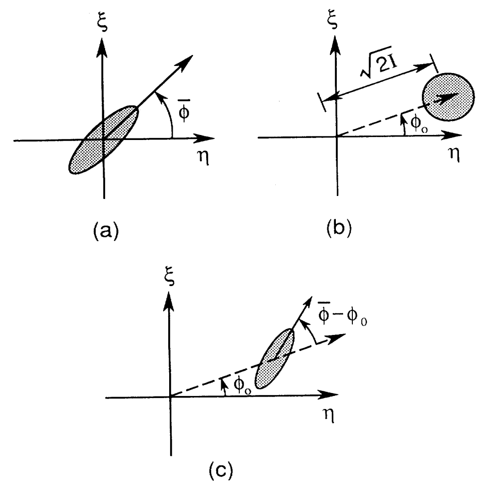

FIG. 1. (a) Mismatched beam injected on axis. (b) Matched beam injected off axis. (c) Mismatched beam injected off axis.

with

$$u(t)=b(t)+c(t)\ , \quad v(t)=b(t)-c(t)\ . \tag{9}$$

The function $d(t)$ has to be determined by the normalization condition of the distribution function. This is done in Appendix A:

$$\int\int d\phi\, dI\, \Psi=1=>d(t)=\frac{1}{\sqrt{b^2(t)-c^2(t)}} =\frac{1}{\sqrt{u(t)v(t)}}\ . \tag{10}$$

We introduce Eq. (8) into Eq. (3), perform the partial differentiation, and order the resulting equation in terms of the canonical variables and their combinations:

$$\begin{aligned}
&\text{Constant:}\quad -\tau d'(t)/d(t)=2-2\sigma b(t)\ ,\\
&I:\quad -\tau b'(t)=-2b(t)+2\sigma(b^2(t)+c^2(t))\ ,\\
&I\cos(2\Omega):\quad -\tau c'(t)=-2c(t)+4\sigma b(t)c(t)\ ,\\
&\cos(2\Omega):\quad 0=-2c(t)\sigma+2c(t)\sigma\ ,\\
&I\cos^2(2\Omega):\quad 0=2c^2(t)\sigma-2c^2(t)\sigma\ ,
\end{aligned} \tag{11}$$

where the prime denotes differentiation with respect to t. The fourth and fifth of the relations in Eq. (11) are already fulfilled. The first relation follows from the second and third relations using normalization condition Eq. (10). The remaining set of two differential equations in $b(t),c(t)$ can be solved by introducing the functions u,v defined in Eq. (9):

$$\tau\begin{bmatrix}u'\\ v'\end{bmatrix}=2\begin{bmatrix}u\\ v\end{bmatrix}-2\sigma\begin{bmatrix}u^2\\ v^2\end{bmatrix}\ . \tag{12}$$

These two equations are of Riccati's type. The solution is given by

$$\begin{aligned}
u(t)&=\frac{1}{(1-u_0\exp\{-2t/\tau\})\sigma}\ ,\\
v(t)&=\frac{1}{(1-v_0\exp\{-2t/\tau\})\sigma}\ ,
\end{aligned} \tag{13}$$

where u_0 and v_0 are integration constants. We use the initial condition for $b(t=0)=b_0$ and $c(t=0)=c_0$ in Eq. (7) and determine u_0,v_0 as

$$\begin{array}{lcl}
\text{Longitudinal} & & \text{Transverse}\\
1-\sigma_{z0}^2/\sigma_{z\infty}^2 & \Longleftarrow u_0 \Longrightarrow & 1-1/(\epsilon_{x\infty}b_0-\epsilon_{x\infty}\sqrt{b_0^2-1/\epsilon_{x0}^2})\\
1-\sigma_{\epsilon0}^2/\sigma_{\epsilon\infty}^2 & \Longleftarrow v_0 \Longrightarrow & 1-1/(\epsilon_{x\infty}b_0+\epsilon_{x\infty}\sqrt{b_0^2-1/\epsilon_{x0}^2})
\end{array} \tag{14}$$

These relations are more transparent in the longitudinal phase space since we restricted the initial distribution to an untilted ellipse in phase space. For $\sigma_{z0}/\sigma_{z\infty}=\sigma_{\epsilon0}/\sigma_{\epsilon\infty}$ or $g=1$, the functions $u(t),v(t)$ become equal and the distribution function no longer depends on the angle variable ϕ.

Using Eqs. (13), (14), and (7), we obtain in the transverse plane the necessary condition for $c(t)=0$:

$$b_0=\frac{1}{\epsilon_{x0}}\Longrightarrow 1 =\frac{1}{2}\left[\frac{\beta_0}{\beta}+\frac{\beta}{\beta_0}+\frac{\beta}{\beta_0}\left[\alpha_0-\frac{\beta_0}{\beta}\alpha\right]^2\right]\equiv\beta_{\rm mag}\ . \tag{15}$$

The combination of Twiss parameters on the right-hand side, $\beta_{\rm mag}$, is known as the β-magnification factor [6–8].

The functions $u(t),v(t)$ approach the same equilibrium value: $u(t\to\infty),v(t\to\infty)=1/\sigma$. Furthermore, it follows that $u(t)$ is monotonic, increasing (decreasing) if u_0 is negative (positive). The same statement holds for $v(t)$. The function $c(t)=(u-v)/2$ will therefore tend to zero, and the distribution function at equilibrium becomes independent of the angle variable ϕ.

B. Mismatched beam injected off axis

Up to this point we have assumed that the center of mass of the distribution is injected at the origin of the phase space (on axis), and will remain there throughout the damping process. From Fig. 1(c), it is clear that the off-centered distribution induces an additional angle ϕ dependence in the distribution function which will persist even if the injected beam is matched.

We denote the position of the injected center of mass by (ϵ_0,z_0) or (x_0,p_0). In phase space (ξ,η), we obtain the position of the injected center of mass as

Longitudinal		Transverse
$\epsilon_0\sqrt{\alpha R/\nu_{s0}}$	$\Longleftarrow\hat{\xi}(t=0)\Longrightarrow$	$x_0/\sqrt{\beta}$
$z_0\sqrt{\nu_{s0}/\alpha R}$	$\Longleftarrow\hat{\eta}(t=0)\Longrightarrow$	$(\alpha x_0+\beta p_0)/\sqrt{\beta}$

(16)

A natural way to take into account the off-axis injection is by shifting the canonical variables

$$\xi\rightarrow\xi-\hat{\xi}(t)\ ,$$
$$\eta\rightarrow\eta-\hat{\eta}(t)\ ,$$

where the functions $\xi(t)$ and $\eta(t)$ have to satisfy the damped oscillator equation associated to the Fokker-Planck equation (3), with the initial condition given by Eq. (16). The corresponding substitution in action-angle variables might look like

$$\begin{aligned}\sqrt{I}\ \cos(\Omega)&\rightarrow\sqrt{I}\ \cos(\Omega)-\sqrt{\hat{I}(t)}\cos(\phi_0-\bar{\phi})\ ,\\ \sqrt{I}\ \sin(\Omega)&\rightarrow\sqrt{I}\ \sin(\Omega)-\sqrt{\hat{I}(t)}\sin(\phi_0-\bar{\phi})\ ,\end{aligned}\tag{17}$$

where Ω is defined in Eq. (8), and $\hat{I}(t)$ and the constant ϕ_0 are related to the initial values of $[\hat{\xi}(0),\hat{\eta}(0)]$,

$$\hat{I}(0)=\tfrac{1}{2}[\hat{\xi}^2(0)+\hat{\eta}^2(0)]\quad\text{and}\quad\tan(\phi_0)=\hat{\xi}(0)/\hat{\eta}(0)\ .\tag{18}$$

We introduce the substitution rules of Eq. (17) into the distribution function Eq. (9)

$$\begin{aligned}\Psi_0(t)=\frac{1}{2\pi d(t)}\exp\{&-u(t)[\sqrt{I}\ \cos(\Omega)-\sqrt{\hat{I}(t)}\cos(\Omega_0)]^2\\ &-v(t)[\sqrt{I}\ \sin(\Omega)-\sqrt{\hat{I}(t)}\sin(\Omega_0)]^2\}\ ,\end{aligned}\tag{19}$$

with

$$\Omega=\phi-\omega t-\bar{\phi}\ ,\quad \Omega_0=\phi_0-\bar{\phi}\ .$$

The distribution function in Eq. (19) has to satisfy Eq. (3). Following Sec. I A quite closely, we perform the partial differentiation in Eq. (3), and order the result in terms of canonical variables and their linear-independent combinations. Thus we obtain the functional dependence of $\hat{I}(t)$:

$$\hat{I}(t)=\hat{I}(0)\exp\{-2t/\tau\}\ ,\tag{20}$$

and $\hat{I}(0)$ is given by Eq. (18). The normalization function $d(t)$ and the functional dependence of $u(t),v(t)$ remain unchanged with respect to the case of on-axis injection, and are given by Eqs. (10), (13), and (14). A result similar to Eq. (19) has been obtained by Chandrasekhar in the analysis of Brownian motion bounded by a quadratic potential [9].

II. DISTRIBUTION FUNCTION IN THE PRESENCE OF NONLINEAR FIELDS

Nonlinear fields will induce a tune spread in the bunch population and, as a consequence, cause the injected emittance to filament [10]. When injected off axis, the center-of-mass position observed with a beam position monitor will be seen to decohere [11]. This effect is not particular to the injection of electron rings. Decoherence was used in proton rings to study the influence of higher-order multipole fields on the beam [12,13].

A convenient way to deal with nonlinear fields is to introduce action-angle variables and to average the perturbation over the fast-evolving variable [14]. This averaged Hamiltonian is now a function of the action variable only, and the tune depends on the action variable:

$$\begin{aligned}H(I)&=\frac{\nu_0}{R}(I-\tfrac{1}{2}uI^2)\ ,\\ \nu(I)&=R\frac{dH(I)}{dI}=\nu_0(1-\mu I)\ .\end{aligned}\tag{21}$$

In the longitudinal plane, μ originates from the expansion of the rf wave with respect to the longitudinal position, in the transverse case from octopole fields. From Ref. [15], we have

Longitudinal		Transverse
$h^2\alpha/8R\nu_{s0}$	$\Longleftarrow\mu\Longrightarrow$	$-(1/16\nu_x\pi)\oint ds\,\beta^2(s)K_3(s)$

(22)

where h denote the harmonic number, and $K_3(s)$ contains the distribution of magnetic octupoles around the ring. The Fokker-Planck equation is given by

$$\tau\Psi_t=2\Psi+2(I+\sigma)\Psi_I=2\sigma I\Psi_{II}-\tau\omega(I)\Psi_\phi+\tfrac{1}{2}\sigma\frac{1}{I}\Psi_{\phi\phi}\ ,\tag{23}$$

with $\omega(I)=2\pi\nu(I)/T_0$, where T_0 denotes the time for one revolution and σ as defined in Eq. (4). Using the Hamilton-Jacobi perturbation method, one may derive an additional contribution to μ that originates from the sextupole distribution around the ring. In this case, the action-angle variables have to be transformed from (ϕ,I) to (ϕ',I') [16,17]. However, the treatment of the Fokker-Planck equation in the canonical variables (ϕ',I') would be a great deal more complicated.

We consider on-axis injection, and try to approach the solution with a test function that is very close to Eq. (8):

$$\Psi(t)=\frac{1}{2\pi d(t)}\exp\{-I[b(t)+c(t)\cos(2\Omega)]\}\ ,\tag{24}$$

with

$$\Omega=\phi+h(t)+f(t)I-\bar{\phi}\ .$$

Note that Ω now contains the action variable, and $h(t)$

and $f(t)$ are yet unknown functions. We next insert Eq. (24) into Eq. (23), and order the resulting equation in terms of canonical variables. We obtain the same set of differential equations that had been derived for the linear case of Eq. (11), plus five additional terms;

$$I\sin(2\Omega)c(t)[\tau h'(t)+\tau\omega_0-6\sigma f(t)]\ , \tag{25}$$

$$I^2\sin(2\Omega)c(t)[-\tau f'(t)+\tau\omega_0\mu+2f(t)-4\sigma b(t)f(t)]\ , \tag{26}$$

$$I^2\sin(4\Omega)\sigma c^2(t)f(t)\ , \tag{27}$$

$$I^3\sin^2(2\Omega)\sigma c^2(t)f^2(t)\ , \tag{28}$$

$$I^2\cos(2\Omega)\sigma c(t)f^2(t)\ , \tag{29}$$

where $b(t)$ and $c(t)$ are given by the corresponding functions of the linear case in Sec. I. We will now show that, under certain assumptions which apply for a damping ring, the terms in Eqs. (27)–(29) are small compared to the other terms. We mentioned in the discussion at the end of Sec. I A that the function $c(t)$ goes to zero as t approaches infinity. The initial value $c(t)=c_0$ is known from Eq. (7) to be in the order of $1/\sigma_0$, and σ_0 was defined previously in Eq. (4). We estimate the magnitude of $c(t)$, $b(t)$, and I, which will be different at injection than at equilibrium:

	Injection	Equilibrium
$c(t)\sim$	$1/\sigma_0$	0
$b(t)\sim$	$1/\sigma_0$	$1/\sigma$
$I\sim$	σ_0	σ

(30)

We are able to estimate the magnitude of the different terms in Eq. (26):

$$\tau f'(t)-2f(t)+4\sigma b(t)f(t)=\tau\omega_0\mu\ ,$$

where all terms on the left-hand side contribute with the same magnitude and $f(t)$ will be of the order of the right-hand side: $f(t)\sim\tau\omega_0\mu\sim\mu$. In the case of Eq. (25) we have

$$\tau h'(t)=-\tau\omega_0-6\sigma f(t)\ ,$$

and $h(t)$ is in the order of $\tau\omega_0\sim 1$. For the five terms in Eqs. (25)–(29), we may summarize their order of magnitude:

				Injection	Equilibrium
I	$\sin(2\Omega)$	$c(t)\tau h'(t)$	$\sim$	1	0
I^2	$\sin(2\Omega)$	$c(t)\tau f'(t)$	$\sim$	$\sigma_0\mu$	0
σI^2	$\sin(4\Omega)$	$c^2(t)f(t)$	$\sim$	$\sigma\mu$	0
σI^3	$\sin^2(2\Omega)$	$c^2(t)f^2(t)$	$\sim$	$\sigma_0\sigma\mu^2$	0
σI^2	$\cos(2\Omega)$	$c(t)f^2(t)$	$\sim$	$\sigma_0\sigma\mu^2$	0

Damping rings operate by definition in the regime $\sigma_0\gg\sigma$. With this assumption, we keep only terms of the orders 1 and $\mu\sigma_0$, and neglect all other terms of the orders $\sigma\mu$ and $\sigma_0\sigma\mu^2$. Since we ignored only terms containing $\mu\sigma$, it is clear that the solution will be exact in the limit of no quantum excitation. Furthermore, the solution will reproduce the distribution function of the linear problem with $\mu=0$.

The functions $f(t)$ and $h(t)$ are thus given by the differential equations

$$\tau f'(t)-2f(t)+4\sigma b(t)f(t)=\tau\omega_0\mu\ , \tag{31}$$

$$\tau h'(t)=-\tau\omega_0\ . \tag{32}$$

Both functions have to satisfy the initial condition $h(0)=f(0)=0$. As a solution for $f(t)$, we find

$$f(t)=\tfrac{1}{2}\omega_0\mu\tau\frac{\exp\{2t/\tau\}-2(u_0+v_0)t/\tau-u_0v_0\exp\{-2t/\tau\}-1+u_0v_0}{\exp\{2t/\tau\}-u_0-v_0+u_0v_0\exp\{-2t/\tau\}}\ , \tag{33}$$

with u_0 and v_0 defined in Eq. (14). By integrating Eq. (32), we find $h(t)=-\omega_0 t$, and Ω in Eq. (24) is given by

$$\Omega=\phi-\omega_0 t+f(t)I-\bar{\phi}\ . \tag{34}$$

A particularly important role will be played by the function $f(t)$, since it is the driving term for the filamentation process. Shortly after injection, i.e., $t\ll\tau$, $f(t)$ behaves like $\omega\mu t$ and increases linearly with time. Then, after the damping process, $f(t)$ approaches the limit, $f(t\to\infty)=\omega\mu\tau/2$. The functions $b(t)$, $c(t)$, and $d(t)$ are tied via Eqs. (9) and (10) to $u(t)$ and $v(t)$, which are given in Eqs. (13) and (14). We mentioned previously that the distribution function in Eq. (24) is an exact solution to the Fokker-Planck equation, if we neglect quantum excitation. In this limit, u_0,v_0 goes to infinity, and the function $f(t)$ becomes

$$f(t)=\tfrac{1}{2}\omega_0\mu\tau(\exp\{2t/\tau\}-1)\ ,$$

for no quantum excitation . (35)

It should be stressed that the distribution function in Eq. (24) will lose its phase dependence in the limit of $t \to \infty$ as $c(t)$ approaches zero. Furthermore, it follows from Eq. (30) that the equilibrium distribution will be Gaussian and independent of μ. On the other hand, it is well known that nonlinear fields will affect the equilibrium distribution. This was shown, for example, in Ref. [5] using the canonical variables ξ,η and solving the Fokker-Planck equation with $\Psi_t=0$.

In our approach, which is based on action-angle variables and an averaged Hamiltonian, we lose this asymptotic characteristic of the distribution function. This is probably the price we have to pay in order to obtain the explicit time dependence of the distribution function.

So far, we have considered the injected distribution to be centered on the closed orbit. In a similar way, we may derive an approximated solution to the Fokker-Planck equation for an off-axis-injected distribution. The distribution function

$$\Psi(t)=\frac{1}{2\pi d(t)}\exp\{-u(t)[\sqrt{I}\,\cos(\Omega)-\sqrt{\hat{I}(t)}\cos(\Omega_0)]^2-v(t)[\sqrt{I}\,\sin(\Omega)-\sqrt{\hat{I}(t)}\sin(\Omega_0)]^2\}\ , \tag{36}$$

with

$$\Omega=\phi-\omega t+f(t)I-\bar{\phi}\ ,\quad \Omega_0=\phi_0-\bar{\phi}\ ,$$

$$\hat{I}(t)=\hat{I}(0)\exp\left\{\frac{-2t}{\tau}\right\}\ ,$$

and $f(t)$ given by Eq. (33) satisfies the Fokker-Planck equation, if we again neglect terms of the order $\mu\sigma$. The functions $u(t),v(t)$ and the normalization function $d(t)$ are defined in Sec. 1(a), and are not affected by the nonlinear terms in the Hamiltonian. This is not surprising: the normalization function $d(t)$ corresponds to the area of the beam ellipsoid, which should remain constant in the absence of damping and quantum excitation, as required by Liouville's theorem. Hence, nonlinear terms in the Hamiltonian cannot affect the area of the evolution of the injected beam ellipsoid.

III. VARIOUS MOMENTS OF THE DISTRIBUTION FUNCTION

By virtue of the relatively simple algebraic form of the distribution function, we may evaluate first and second moments. In Appendix A, the different moments of the mismatched and centered distribution function are derived. It turns out that the odd moments will vanish because of the symmetry: $\Psi(I,\phi,t)=\Psi(I,\phi+\pi,t)$. We want to compare the analytic formula of the second moment $\langle z^2\rangle$ with multiparticle simulation.

First we discuss the multiparticle simulation. The one-turn map in longitudinal phase space includes radiation damping and quantum excitation (QE), and consists of three steps:

$$\Delta z=-\alpha\epsilon\ ,\quad \text{over the ring}$$

$$\Delta\epsilon=-(eV_{\mathrm{RF}}/E_0)\{\sin[\phi_s-(h/R)z]-\sin(\phi_s)\}\ ,\quad \text{rf cavity}\ ;$$

$$\Delta\epsilon=-\lambda\epsilon+\sigma_{\epsilon\infty}\sqrt{1-\lambda^2}\hat{q}\ ,\quad \text{damping + QE}\ ,$$

where ϕ_s denotes the synchronous phase, $\hat{q}$ is a random Gaussian variable with unit standard deviation, and the damping coefficient is defined by $\lambda=\exp(-2T_0/\tau)$ [18]. One damping time corresponds to about 15 000 revolutions, 3000 particles were tracked over 20 000 turns, and the second moment was calculated after every three turns.

From Appendix A and Eq. (1), we obtain the time evolution of the second moment:

$$\langle z^2\rangle=\frac{\sigma_{\epsilon\infty}}{\sigma_{z\infty}[b^2(t)-c^2(t)]}(b(t)-c(t)\{\cos(2\omega_0 t+2\bar{\phi})\mathrm{Re}[Z^{3/2}(t)]+\sin(2\omega_0 t+2\bar{\phi})\mathrm{Im}[Z^{3/2}(t)]\})\ , \tag{37}$$

with

$$Z(t)=1\Bigg/\left[1-\frac{i4f(t)b(t)+f^2(t)}{b^2(t)-c^2(t)}\right]\ .$$

The comparison between the analytic result and the simulation is shown in Fig. 2. The pictures on the bottom and on the top display data belonging to the same run. On the top picture we see a fairly good agreement within the first 1000 turns. A slight disagreement shows up as a "wiggly" pattern after 2000 turns (bottom pictures). However, this pattern does not originate from the approximations done in Sec. II, since it persists in the case without quantum excitation, where the above equation is an exact solution of the Fokker-Planck equation.

The Hamiltonian formalism is based on *differential* equations and assumes the rf cavity to be spread over the ARC, whereas the mapping used in the simulation consists of *difference* equations. This might be the actual source of the small discrepancy in the bottom pictures of Fig. 2.

Matched beam injected off axis

The phase portrait of this distribution function at injection is displayed in Fig. 1(b). A matched beam implies $c(t)\equiv 0 \rightarrow u(t)=v(t)$, and the distribution function in Eq. (36) simplifies to

$$\Psi(t)=\frac{b(t)}{2\pi}\exp\{-b(t)[I+\hat{I}(t)-2\sqrt{I\hat{I}(t)}\cos\{\phi-\omega t+f(t)I-\phi_0\}]\}\ , \tag{38}$$

with

$$\hat{I}=\hat{I}(0)\exp\{-2t/\tau\}\ .$$

A sufficient condition for the beam to be matched to the lattice in the transverse plane is $\beta_{\text{mag}}=1$ or, equivalently, $b_0=1/\epsilon_{x0}$. In the longitudinal plane, $g=1$ is required. In Appendix B, we derive the first and second moments for the general case of a mismatched and off-centered injection. From Eq. (B3) we obtain, with $c(t)=0$, $A=b(t)$ and $\tilde{\Omega}_0=\Omega_0$,

$$\begin{aligned}\langle\xi\rangle&=\frac{\sqrt{2\hat{I}(t)}}{(1+\theta^2)^2}\exp\left\{-\frac{\theta^2\hat{I}(t)b(t)}{1+\theta^2}\right\}[(1-\theta^2)\sin(\Phi_1)-2\theta\cos(\Phi_1)]\ ,\\ \langle\eta\rangle&=\frac{\sqrt{2\hat{I}(t)}}{(1+\theta^2)^2}\exp\left\{-\frac{\theta^2\hat{I}(t)b(t)}{1+\theta^2}\right\}[(1-\theta^2)\sin(\Phi_1)+2\theta\sin(\Phi_1)]\ ,\end{aligned} \tag{39}$$

with

$$\theta=\frac{f(t)}{b(t)}\quad\text{and}\quad \Phi_1=\omega_0 t+\phi_0-\frac{\theta\hat{I}(t)b(t)}{1+\theta^2}\ .$$

From the discussion in Sec. II, we realize that θ behaves shortly after injection as $\theta(t\ll\tau)=\sigma_0\mu\omega t$, and will increase with time. The quantity θ might be extracted from a given set of beam-position measurements over successive turns after injection. Thus the injected emittance may be measured if the nonlinear perturbation μ is known. The denominator in Eqs. (39) grows with time and causes the decoherence of the center-of-mass motion. After a sufficient number of damping times, θ approaches the limit $\theta(t\gg\tau)=\sigma\mu\omega\tau/2$. At that time, the center-of-mass motion approaches zero, due to $\hat{I}(t\rightarrow\infty)=0$. The second moments are obtained from Eq. (B4)

$$\langle\xi^2\rangle=\frac{1}{b(t)}+\hat{I}(t)\left\{1-\frac{\exp\{-[4\theta^2\hat{I}(t)b(t)]/(1+4\theta^2)\}}{(1+4\theta^2)^3}[(1-12\theta^2)\cos(2\Phi_2)+(6\theta-8\theta^3)\sin(2\Phi_2)]\right\}\ ,$$

$$\langle\eta^2\rangle=\frac{1}{b(t)}+\hat{I}(t)\left\{1+\frac{\exp\{-[4\theta^2\hat{I}(t)b(t)]/(1+4\theta^2)\}}{(1+4\theta^2)^3}[(1-12\theta^2)\cos(2\Phi_2)+(6\theta-8\theta^3)\sin(2\Phi_2)]\right\}\ ,$$

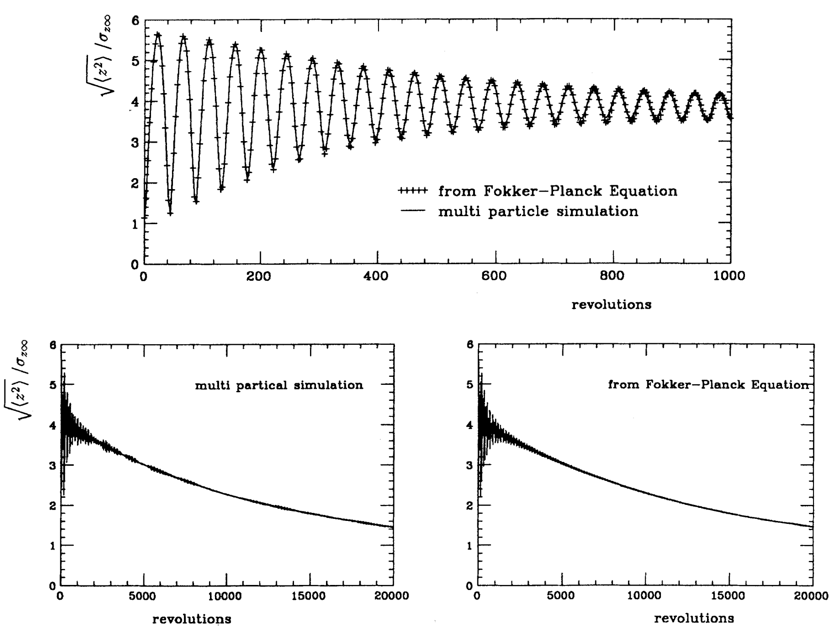

FIG. 2. The time evolution of the bunch length.

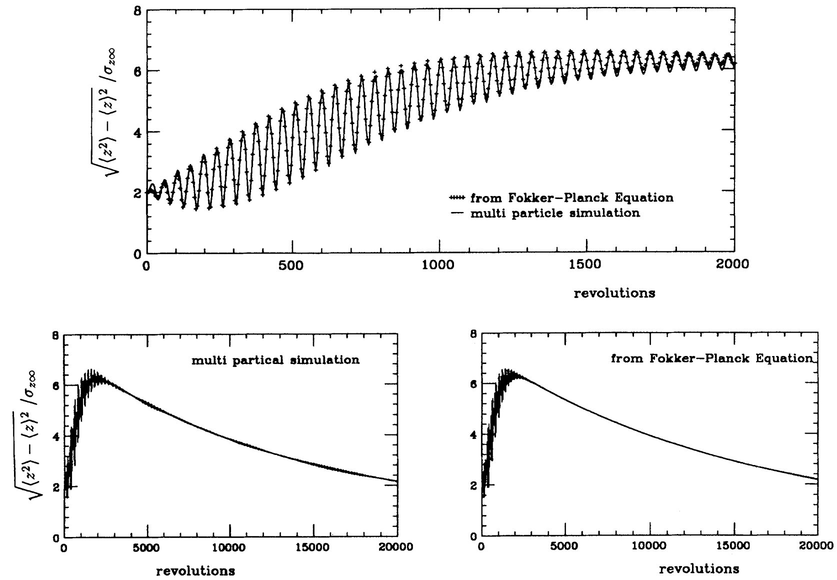

FIG. 3. The time evolution of the bunch after off-axis injection.

and

$$\Phi_2=\omega_0 t+\phi_0-\frac{\theta\hat{I}(t)b(t)}{1+4\theta^2} .$$

At this point, we want to compare the analytic relation for the second moment in the longitudinal plane, $\langle z^2\rangle-\langle z\rangle^2$, to multiparticle simulations which were done with 3000 particles. The data of the first 2000 turns in the pictures on the bottom of Fig. 3 are expanded in the top picture of Fig. 3. The analytic expression is in good agreement with the simulation result. A small deviation within the first synchrotron oscillation is a consequence of the Hamiltonian in action variable $H(I)$, which was obtained by averaging over the phase terms. The bottom pictures show the initial growth of the bunch length due to filamentation. After turn 2500, the bunch length starts to decrease due to radiation damping, and slowly approaches the equilibrium value $\sigma_z(t)/\sigma_{z\infty}=1$.

IV. SUMMARY

In Sec. II we presented an approximate solution to the Fokker-Planck equation that describes the injection process into a storage ring under the influence of nonlinear fields. The explicit time dependence of the first and second moments on the approximate solution of the Fokker-Planck equation was derived, and compared well to results obtained from multiparticle simulations. These simulations included radiation damping and the effect of quantum excitation on the particle trajectory. The analytic result for the first moment of the particle distribution may be used to extract the injected emittance from a set of beam-position measurements over successive turns after injection.

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APPENDIX A: HIGHER-ORDER MOMENTS OF THE CENTERED DISTRIBUTION FUNCTION

In this appendix, we derive higher-order moments of the distribution functions discussed in Secs. I and II:

$$\Psi(t)=\frac{1}{2\pi d(t)}\exp\{-I[b(t)+c(t)\cos(2\Omega)]\} , \tag{A1}$$

with

$$\Omega=\phi-\omega_0 t+f(t)I-\bar{\phi} .$$

It is important to notice the symmetry $\Psi(I,\phi,t)=\Psi(I,\phi+\pi,t)$, which reflects the invariance of the distribution function under the transformation $(\xi,\eta)\rightarrow(-\xi,-\eta)$. As a consequence, all moments of odd order will vanish. What remains are the moments of even order, which will be treated in action-angle variables:

$$\langle\xi^{2m}\rangle=\int\int\xi^{2m}d\xi\,d\eta=2^m\int\int I^m\sin^{2m}(\phi)dI\,d\phi ,$$

and

$$\langle\eta^{2m}\rangle=\int\int\eta^{2m}d\xi\,d\eta=2^m\int\int I^m\cos^{2m}(\phi)dI\,d\phi .$$

Let us evaluate first the expression $\langle\eta^{2m}\rangle$. We use the above expression for the distribution function and obtain

$$\langle\eta^{2m}\rangle=\frac{2^m}{2\pi d}\int_0^\infty dI\,\exp\{-Ib\}R_m(I) , \tag{A2}$$

with

$$R_m(I)=\int_0^{2\pi}\exp\{-Ic\cos(2\Omega)\}\cos^{2m}(\phi) .$$

Next we use the identity [19]

$$\cos^{2m}(\phi)=\frac{1}{2^{2m}}\sum_{k=0}^{2m}\begin{bmatrix}2m\\k\end{bmatrix}\cos(2\phi m-2\phi k)\ . \quad \text{(A3)}$$

We introduce this expression in the definition of $R_m(I)$, integrate with respect to Ω, and obtain the result in terms of Bessel functions:

$$R_m(I)=\frac{2\pi}{2^m}\sum_{k=0}^{2m}\begin{bmatrix}2m\\k\end{bmatrix}J_{m-k}(iIc)\times\cos\{2[\omega_0 t-f(t)I+\bar{\phi}]\}\ , \quad \text{(A4)}$$

where i denotes the imaginary unit. Equation (A2) can now be integrated, and the result contains hypergeometric functions [19]

$$\langle\eta^{2m}\rangle=\frac{(2m)!}{2d(t)}\sum_{l=0}^{m}\frac{1}{(m-l)!l!(1+\delta_{l,0})}\left[\frac{-c(t)}{2}\right]^l\left\{\frac{\exp\{2il[\omega_0 t+\bar{\phi}]\}F\left[\frac{l+m+1}{2},\frac{l-m}{2};l+1,-z_{2l}\right]}{\sqrt{[\beta_{2l}^2(t)-c^2(t)]^{l+m+1}}}+\text{c.c.}\right\}\ ,$$

where c.c. denotes the complex conjugate of the preceding term.

$$\beta_l(t)=b(t)+ilf(t)\ ,\quad z_l=\frac{c^2(t)}{\beta_l^2(t)-c^2(t)}\ , \quad \text{(A5)}$$

and $\delta_{l,0}$ is the Kronecker δ. A similar expression may be derived for the other canonical variable

$$\langle\xi^{2m}\rangle=\frac{(2m)!}{2d(t)}\sum_{l=0}^{m}\frac{1}{(m-l)!l!(1+\delta_{l,0})}\left[\frac{-c(t)}{2}\right]^l\left\{\frac{\exp\{2il[\omega_0 t+\bar{\phi}]\}F\left[\frac{l+m+1}{2},\frac{l-m}{2};l+1,-z_{2l}\right]}{\sqrt{[\beta_{2l}^2(t)-c^2(t)]^{l+m+1}}}+\text{c.c.}\right\}\ ,$$

where a minus at $c^l(t)$ is the only difference from the previous relation. For $m=0$, we obtain the normalization condition that was used earlier in this paper:

$$1\equiv\langle\eta^0\rangle=\frac{1}{d(t)}\frac{1}{\sqrt{b^2(t)-c^2(t)}}=>d(t)=\frac{1}{\sqrt{b^2(t)-c^2(t)}}\ . \quad \text{(A6)}$$

The second moment for $m=1$ describes the evolution of the bunch length, energy spread, or beam size. After some rearrangements, we obtain

$$\langle\eta^2\rangle=\frac{1}{b^2(t)-c^2(t)}(b(t)-c(t)\{\cos[2(\omega_0 t+\bar{\phi})]\text{Re}[Z^{3/2}(t)]+\sin[2(\omega_0 t+\bar{\phi})]\text{Im}[Z^{3/2}(t)]\})\ , \quad \text{(A7)}$$

$$\langle\xi^2\rangle=\frac{1}{b^2(t)+c^2(t)}(b(t)+c(t)\{\cos[2(\omega_0 t+\bar{\phi})]\text{Re}[Z^{3/2}(t)]+\sin[2(\omega_0 t+\bar{\phi})]\text{Im}[Z^{3/2}(t)]\})\ . \quad \text{(A8)}$$

These expressions contain the real and the imaginary part of the following complex function:

$$Z(t)=1\Big/\left[1-\frac{i4f(t)b(t)+f(t)^2}{b^2(t)-c^2(t)}\right]\ .$$

As mentioned before, the function $f(t)$ will increase shortly after injection linearly with time, and $Z(t)$ will act like a damping term. Later, when the beam approaches equilibrium, $t\rightarrow\infty$: $Z(t)$ will also approach a limiting value. With $b(t\rightarrow\infty)=1/\sigma$ and $c(t\rightarrow\infty)=0$, we find

$$Z(t\rightarrow\infty)=\frac{1}{1-i2\omega_0\mu\tau\sigma-(\omega_0^2\mu^2\tau^2\sigma^2/4)}\ .$$

The contribution of $Z(t)$ to the beam size scales with $c(t)$ and will be small as $c(t)$ approaches zero, after a couple of radiation damping times.

APPENDIX B: FIRST AND SECOND MOMENTS OF THE OFF-CENTERED, MISMATCHED DISTRIBUTION FUNCTION

This is the general case shown in Fig. 1(c). Analytic expressions for the first and second moments can be compared to beam-position or beam-size measurements after injection. These expressions are of practical interest in order to understand and optimize the injection process. It turns out that the integrals involved cannot be solved directly by means of integral tables [19,20], and the solution can only be given in a power series containing hypergeometric functions. We start with the distribution function given by Eq. (19):

$$\Psi=\frac{\sqrt{uv}}{2\pi}\exp\{-u[\sqrt{I}\cos(\Omega)-\sqrt{\hat{I}}\cos(\Omega_0)]^2-v[\sqrt{I}\sin(\Omega)-\sqrt{\hat{I}}\sin(\Omega_0)]^2\}\ ,$$

with

$$\Omega=\phi-\omega t-\bar{\phi}\ ,\quad \Omega_0=\phi_0-\bar{\phi}\ .$$

We keep in mind that $\omega t=\omega_0 t-f(t)I$ depends on the action variable. In the exponent, we substitute $u(t),v(t)$ by $b(t),c(t)$ via Eq. (9) and obtain

$$\Psi=\frac{\sqrt{uv}}{2\pi}\exp\{-Ib-\hat{I}[b+c\cos(2\Omega_0)] -Ic\cos(2\Omega)+2\sqrt{I\hat{I}}\,A\cos(\Omega-\tilde{\Omega}_0)\}\ , \tag{B1}$$

with

$$\tan(\tilde{\Omega}_0)=\frac{b-c}{b+c}\tan(\Omega_0)\ ,$$

$$A=\sqrt{b^2+c^2+2bc\cos(2\Omega_0)}\ .$$

The first and second moments lead to the following type of integrals over the angle variable:

$$R(I)=\frac{1}{2\pi}\int_0^{2\pi}F(\phi)\exp\{-Ic\cos(2\Omega)+2\sqrt{I\hat{I}}\,A\cos(\Omega-\tilde{\Omega}_0)\}d\phi\ ,$$

with

$$F(\phi)=\begin{cases}\cos(\phi)\\ \sin(\phi)\\ \dfrac{1\pm\cos(2\phi)}{2}\end{cases}\ .$$

We change the integration variable from ϕ to ζ,

$$R(I)=\frac{1}{2\pi}\int_0^{2\pi}F(\zeta+\omega t+\bar{\phi}+\tilde{\Omega}_0)\times\exp\{-Ic\cos(2\zeta+2\tilde{\Omega}_0)\}\times\{\cos[i2\sqrt{I\hat{I}}\,A\cos(\zeta)]-i\sin[i2\sqrt{I\hat{I}}\,A\cos(\zeta)]\}d\zeta\ .$$

Either the sines or the cosines of the trigonometric function will give a zero contribution in the integration due to symmetry. We expand the remaining trigonometric function in a power series:

$$\cos[i2\sqrt{I\hat{I}}\,A\cos(\zeta)]=\sum_{n=0}^{\infty}\frac{(2\sqrt{I\hat{I}}\,A)^{2n}}{(2n)!}\cos^{2n}(\zeta)\ ,$$

$$\sin[i2\sqrt{I\hat{I}}\,A\cos(\zeta)]=i\sum_{n=0}^{\infty}\frac{(2\sqrt{I\hat{I}}\,A)^{2n+1}}{(2n+1)!}\cos^{2n+1}(\zeta)\ ,$$

and substitute the expression given in Eq. (A3). The integration over the angle ζ results in Bessel functions. The second integration over the action variable leads to a power series containing hypergeometric functions. To simplify the notation in the final expressions, we define

$$G_j^{n,l}=\left[\frac{c(t)}{2\beta_j(t)}\right]^{n-1}\frac{(2n-l+1)!}{\beta_j^{n+2}(t)\Gamma(n-l+1)}F\left\{\frac{2n-l+2}{2},\frac{2n-l+3}{2},n-l+1;\left[\frac{c(t)}{\beta_j(t)}\right]^2\right\}\ , \tag{B2}$$

where $\beta_j(t)$ is defined in Eq. (A5). For the first moments of the distribution, we obtain

$$\langle\eta\rangle+i\langle\xi\rangle=\left[\frac{uv}{2}\right]^{1/2}\exp\{-\hat{I}(t)[b+c\cos(2\Omega_0)]\}\sum_{n=0}^{\infty}\sum_{k=0}^{2n+1}\frac{(-1)^{n-k+1}[A\sqrt{\hat{I}(t)}]^{2n+1}}{k!(2n-k+1)!}\times(\exp\{i[-(2n-2k+1)\tilde{\Omega}_0+\omega_0 t+\bar{\phi}]\}G_1^{n,k-1}-\exp\{i[(2n-2k+1)\tilde{\Omega}_0+\omega_0 t+\bar{\phi}]\}G_1^{n,k})\ . \tag{B3}$$

The result for the second moments is given by

$$\left.\begin{matrix}\langle\xi^2\rangle\\ \langle\eta^2\rangle\end{matrix}\right\}=\sqrt{uv}\exp\{-\hat{I}[b+c\cos(2\Omega_0)]\}\sum_{n=0}^{\infty}\sum_{k=0}^{2n}\frac{(-1)^{n-k}(A^2\hat{I})^n}{k!(2n-k)!}\times[\cos[2(n-k)\tilde{\Omega}_0]G_0^{n,k}\pm\tfrac{1}{4}(\exp\{2i[-(n-k)\tilde{\Omega}_0+\omega_0 t+\bar{\phi}]\}G_2^{n,k-1}+\exp\{2i[(n-k)\tilde{\Omega}_0+\omega_0 t+\bar{\phi}]\}G_2^{n,k+1}+\text{c.c.})]\ , \tag{B4}$$

where c.c. denotes the complex conjugate of the preceding terms. The above relations considerably simplify in the case of a centered injection with $\hat{I}(t)=0$ or, in the case of a matched injection, with $c(t)=0$.

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